

# Reduction of $su(N)$ loop tensors to trees

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February 7, 2008

## Abstract

We present a systematic method to express all  $su(N)$  invariant tensors in terms of forests i.e. products of tree tensors.

## 1 Introduction

It is a well known fact that in a simple Lie algebra of rank  $r$  there are exactly  $r$  independent Casimir invariants. This causes a direct restriction on algebra of  $su(N)$  tensors since naively one could produce an infinite amount of invariants by contracting d,f tensors. The Cayley-Hamilton theorem which is the reason of why higher Casimir invariants are dependant on the first  $r$  ones clearly gives additional  $d_{ijk}$  tensor identities. The systematic, computer friendly, approach to obtain these formulas was presented by Sudbery [1]. One may also define matrices  $[F_i]_{jk} = f_{ijk}$ ,  $F = a_i F_i$ ,  $a_i \in \mathbf{C}$  and use Cayley-Hamilton equation to obtain analogous identities for  $f_{ijk}$  which was elaborated in details in [2]. In this paper we will use a geometrical approach to find formulas on  $su(N)$  loop tensors in term of  $su(N)$  tree tensors. In this way we give a recursive method which allows to express any  $su(N)$  invariant tensor in terms of basic ones i.e. forests ( products of trees ). In section 3 we prove several lemmas and eventually the main result. In section 4 we present a few examples to give the insight into the method.

We will use the following conventions

$$\lambda_i \lambda_j = \frac{2}{N} \delta_{ij} \mathbf{1} + d_{ijk} \lambda_k + i f_{ijk} \lambda_k, \quad (1)$$

where  $\lambda_i$ 's are  $su(N)$  generators in fundamental representation and  $d_{ijk}$ ,  $f_{ijk}$  are completely symmetric/antisymmetric structure tensors. Multiplication law (1) together with Jacobi identities for  $\lambda_i$ 's give identities known long time ago [3]. We will make a special use of

$$f_{i_1 i_2 k} d_{k i_3 i_4} + f_{i_1 i_3 k} d_{k i_2 i_4} + f_{i_1 i_4 k} d_{k i_2 i_3} = 0, \quad (2)$$

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and

$$f_{i_1 i_2 k} f_{k i_3 i_4} = \frac{2}{N} (\delta_{i_1 i_3} \delta_{i_2 i_4} - \delta_{i_1 i_4} \delta_{i_2 i_3}) + d_{i_1 i_3 k} d_{k i_2 i_4} - d_{i_1 i_4 k} d_{k i_2 i_3}. \quad (3)$$

## 2 Bird tracks

In order to grasp the variety of all possible invariant tensors it is helpful to introduce the diagrammatic notation for d and f tensors (figure 1). Each leg corresponds to one index and summing over any two indices is simply gluing appropriate legs. This notation is very convenient because now any tensor may be represented by a graph.

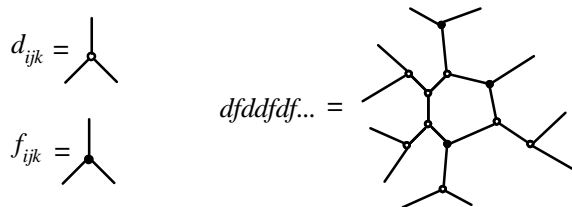


Figure 1:  $d_{ijk}$ ,  $f_{ijk}$  diagrams and a typical tensor diagram.

Such diagrammatic approach has already been introduced long time ago by Cvitanovič. The d, f tensors are called bird tracks since they look like tracks of a bird. The reader is referred to [4] where a vast amount of group properties is rediscovered in such diagrammatic language. Since  $d_{ijk}$  is totally symmetric the order of corresponding legs is irrelevant. For  $f_{ijk}$  we have to set e.g. anticlockwise convention. A special group of diagrams are loop and tree diagrams (figure 2)

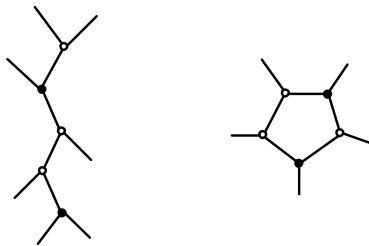


Figure 2: A tree and a loop diagram.

One may rotate any diagram on the plane without changing the value of the corresponding tensor. Reflections ( or rotations in three dimensions ) are allowed as well, however in this case one has to take care of the sign since f tensor is antisymmetric. If a diagram consists only of d tensors then reflections will not affect its value. Several definitions are now in order.

The index that corresponds to d/f tensor is called d/f index. A loop  $L_1$  is smaller then loop  $L_2$  if the number of f,d tensors in  $L_2$  is smaller then the number of f,d tensors on  $L_1$ . Note

that, in general, trees can be attached to loops. In that case we will call it a tree loop diagram. Similarly a tree loop  $L_1$  is smaller then tree loop  $L_2$  if the number of f,d tensors within loop in  $L_2$  is smaller then the number of f,d tensors within loop in  $L_1$ .

A loop diagram is called  $n$  loop if it consists of  $n$  tensors. A loop diagram is called  $d$  loop if it consists of  $d$  tensors only. A loop diagram is called  $1f/2f$  loop if it consists of one/two  $f$  tensor and  $d$  tensors.

### 3 Loop reduction

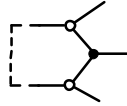
This section consists of several lemmas and eventually a theorem which gives a computational method for expressing loops by trees.

**Lemma 1.** Any loop is a linear combination of  $d$  loops and  $1f$  loops.

*Proof.* Let us rewrite (3) in diagrammatic notation <sup>1</sup>

$$\begin{array}{c} 4 \quad 3 \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \bullet \\ \diagup \quad \diagdown \\ 1 \quad 2 \end{array} = \frac{2}{N} \left( \begin{array}{c} 3 \quad 4 \\ \parallel \quad \parallel \\ 1 \quad 2 \end{array} - \begin{array}{c} 4 \quad 3 \\ \parallel \quad \parallel \\ 1 \quad 2 \end{array} \right) + \begin{array}{c} 2 \quad 4 \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \bullet \\ \diagup \quad \diagdown \\ 1 \quad 3 \end{array} - \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \bullet \\ \diagup \quad \diagdown \\ 1 \quad 4 \end{array} \quad (*)$$

therefore it is sufficient to consider loops where  $f$  tensor is between  $d$  tensors



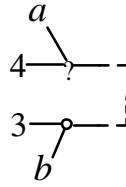
However in such case one can use Jacobi identities (2)

$$\begin{array}{c} 3 \quad 4 \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \bullet \\ \diagup \quad \diagdown \\ 1 \quad 2 \end{array} + \begin{array}{c} 2 \quad 4 \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \bullet \\ \diagup \quad \diagdown \\ 1 \quad 3 \end{array} + \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \bullet \\ \diagup \quad \diagdown \\ 1 \quad 4 \end{array} = 0 \Rightarrow \begin{array}{c} 3 \quad 4 \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \bullet \\ \diagup \quad \diagdown \\ 1 \quad 2 \end{array} + \begin{array}{c} 2 \quad 4 \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \bullet \\ \diagup \quad \diagdown \\ 1 \quad 3 \end{array} - \begin{array}{c} 1 \quad 4 \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \bullet \\ \diagup \quad \diagdown \\ 2 \quad 3 \end{array} = 0 \quad (**)$$

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<sup>1</sup>Instead of writing  $i_1, i_2, i_3, i_4$  ect. we prefer 1, 2, 3, 4 since it causes no misunderstanding and gives a better idea of the structure of indices. Take attention of the order of indices 3 and 4. There is no mistake. The whole diagram is supposed to be read with the anticlockwise convention.

therefore attaching <sup>2</sup>



we get

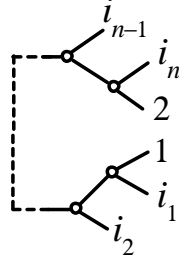
The last identity means that we can "move" f tensor along the loop producing a smaller tree loop. Eventually such f tensor will "meet" another f tensor ( if there is another one in the loop ) and one can use (\*) again to get rid of f tensors. This procedure stops on d loops or 1f loops.  $\square$

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<sup>2</sup>The question mark means that there may be f tensor or d tensor.

**Lemma 2.** Let A be a d loop or 1f loop. Then any permutation of d indices of A does not change the value of A up to trees and smaller tree loops.

*Proof.* Consider identity (\*) and attach the tensor



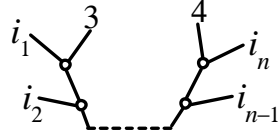
where the dashed lines correspond to d tensors only.  
The result is

$$\begin{array}{c}
 \begin{array}{c}
 \begin{array}{c}
 \text{Diagram 1: } n+1 \text{ loop} \\
 \text{A diagram with a central vertex connected to four other vertices. The top two vertices have lines labeled 4 and 3. The bottom two vertices have lines labeled i_1 and i_n. The left and right vertices have lines labeled i_2 and i_{n-1}. A dashed line connects the two bottom vertices.} \\
 n+1 \text{ loop}
 \end{array}
 & = \frac{2}{N} ( & \begin{array}{c}
 \text{Diagram 2: } n+2 \text{ loop} \\
 \text{A diagram with a central vertex connected to four other vertices. The top two vertices have lines labeled 4 and 3. The bottom two vertices have lines labeled i_2 and i_1. A dashed line connects the two top vertices.} \\
 n+2 \text{ loop}
 \end{array}
 & - & \begin{array}{c}
 \text{Diagram 3: } n+2 \text{ loop} \\
 \text{A diagram with a central vertex connected to four other vertices. The top two vertices have lines labeled i_n and i_{n-1}. The bottom two vertices have lines labeled i_2 and i_1. A dashed line connects the two top vertices.} \\
 n+2 \text{ loop}
 \end{array}
 & ) + & \begin{array}{c}
 \text{Diagram 4: } n+2 \text{ loop} \\
 \text{A diagram with a central vertex connected to four other vertices. The top two vertices have lines labeled i_n and 4. The bottom two vertices have lines labeled 3 and i_1. A dashed line connects the two top vertices.} \\
 n+2 \text{ loop}
 \end{array}
 & - & \begin{array}{c}
 \text{Diagram 5: } n+2 \text{ loop} \\
 \text{A diagram with a central vertex connected to four other vertices. The top two vertices have lines labeled i_n and 3. The bottom two vertices have lines labeled 4 and i_1. A dashed line connects the two top vertices.} \\
 n+2 \text{ loop}
 \end{array}
 \end{array}
 \end{array}$$

Therefore the permutation of indices 3, 4 does not change the d loop up to trees and smaller tree loops. Since indices 3, 4 are not distinguished we can do any permutation of any two indices and the d loop will not change the value up to trees and smaller tree loops. Since any permutation is a proper composition of transpositions the Lemma 2 follows for d loops. The proof for 1f loops is analogous.  $\square$

**Lemma 3.** Any 1f loop is a linear combination of trees and smaller tree loops.

*Proof.* Consider the Jacobi identities (\*\*\*) and attach the tree diagram ( consisting of d tensors only )



The result is

$$+ \quad + \quad = 0$$

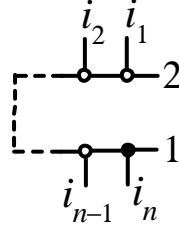
From Lemma 2 it follows that

$$+ \quad + \quad trees + \quad smaller \quad = 0$$

$n+1 \text{ loop} \qquad n+2 \text{ loop}$

**Lemma 4.** Any d loop is a linear combination of trees and smaller tree loops.

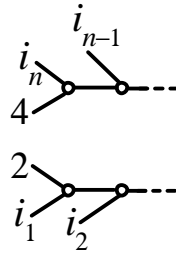
*Proof.* Consider Jacobi identity (\*\*) and attach the following tree ( consisting of one f tensor and  $n - 1$  d tensors )



The result is

$$\begin{array}{ccc}
 \begin{array}{c} \text{Diagram 1: } n+2 \text{ loop} \\ \text{Two horizontal nodes connected by a dashed line. Top node has incoming lines } i_2, i_1 \text{ and outgoing line } 4. \text{ Bottom node has incoming lines } i_{n-1}, i_n \text{ and outgoing line } 3. \end{array} & + & \begin{array}{c} \text{Diagram 2: } n+2 \text{ loop} \\ \text{Two horizontal nodes connected by a dashed line. Top node has incoming lines } i_2, i_1 \text{ and outgoing line } 3. \text{ Bottom node has incoming lines } i_{n-1}, i_n \text{ and outgoing line } 4. \end{array} \\
 & & = & \begin{array}{c} \text{Diagram 3: } n+1 \text{ loop} \\ \text{A tree structure with a central node connected to four other nodes. The top two nodes have incoming lines 3 and 4. The bottom two nodes have incoming lines } i_1, i_2 \text{ and } i_n, i_{n-1}. \end{array} \quad (A)
 \end{array}$$

Therefore the symmetrization of indices 3,4 in such 2f loop is equal to smaller tree loop. Now for the proof of Lemma 4 consider identity (\*) and attach the tree tensor ( consisting of d tensors only )



The result is

$$\begin{array}{c} \text{Diagram 1} \end{array} = \frac{2}{N} \left( \begin{array}{c} \text{Diagram 2} - \text{Diagram 3} \right) + \begin{array}{c} \text{Diagram 4} - \text{Diagram 5}
 \end{array}$$

The diagrams are as follows:

- Diagram 1:** A graph with 6 vertices and 7 edges. It consists of a central square with two additional vertices connected to the top and bottom vertices of the square. The edges are labeled with indices  $i_1, i_2, i_n, i_{n-1}$  and the vertices are labeled 1, 3.
- Diagram 2:** A graph with 6 vertices and 7 edges, similar to Diagram 1 but with different edge labels.
- Diagram 3:** A graph with 6 vertices and 7 edges, similar to Diagram 1 but with different edge labels.
- Diagram 4:** A graph with 6 vertices and 7 edges, similar to Diagram 1 but with different edge labels.
- Diagram 5:** A graph with 6 vertices and 7 edges, similar to Diagram 1 but with different edge labels.

Or simply

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} + \text{trees} + \text{smaller tree-loops}$$

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- Diagram 2:** A graph with 6 vertices and 7 edges, similar to Diagram 1 but with different edge labels. Below it is the label  $n+2 \text{ loop}$ .

According to (A) the symmetrization over indices 1 and  $i_1$  gives

$$\begin{array}{c} \text{Diagram 1} \end{array} + \begin{array}{c} \text{Diagram 2} \end{array} = \text{trees} + \text{smaller tree-loops}$$

The diagrams are as follows:

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- Diagram 2:** A graph with 6 vertices and 7 edges, similar to Diagram 1 but with different edge labels. Below it is the label  $n+2 \text{ loop}$ .

Due to Lemma 2 the Lemma 4 follows.  $\square$



**Theorem.** Any loop diagram is a linear combination of forests.

*Proof.* From Lemma 1 it is sufficient to consider d loops and 1f loops. From Lemma 3 and Lemma 4 we may recursively reduce 1f loop and d loop to arbitrary small loops and ultimately to trees.  $\square$

**Corollary 1.** Any diagram is a linear combination of forests.

*Proof.* Any loop in the diagram may be replaced by a linear combination of trees. This will in general produce more loops however the number of d,f tensors will be smaller after such replacement. Following the induction with respect to the number of d,f tensors we finely reduce all loops.  $\square$

**Corollary 2.** Any diagram is a linear combination of products of trace tensors  $Tr(\lambda_{i_1} \dots \lambda_{i_n})$  where  $\lambda_i$ 's are  $su(N)$  Gell-Mann matrices.

*Proof.* According to Corollary 1 it is sufficient to consider tree diagrams. With help of (1) we have

$$Tr(\lambda_{i_1} \dots \lambda_{i_n}) = \frac{1}{N} Tr(\lambda_{i_1} \lambda_{i_2}) Tr(\lambda_{i_3} \dots \lambda_{i_n}) + (d_{i_1 i_2 k} + i f_{i_1 i_2 k}) Tr(\lambda_k \lambda_{i_3} \dots \lambda_{i_n})$$

therefore

$$d_{i_1 i_2 k} Tr(\lambda_k \lambda_{i_3} \dots \lambda_{i_n}) = \frac{1}{2} Tr(\lambda_{i_1} \lambda_{i_2}) Tr(\lambda_{i_3} \dots \lambda_{i_n}) - \frac{1}{N} Tr(\lambda_{i_1} \lambda_{i_2}) Tr(\lambda_{i_3} \dots \lambda_{i_n})$$

and

$$f_{i_1 i_2 k} Tr(\lambda_k \lambda_{i_3} \dots \lambda_{i_n}) = \frac{1}{2} Tr(\lambda_{[i_1} \lambda_{i_2]}) Tr(\lambda_{i_3} \dots \lambda_{i_n})$$

hance Corollary 2 follows by induction .  $\square$

## 4 Examples

Below we give  $su(N)$  formulae for the lowest d loops i.e. triangles, squares and pentagons. The identities for triangles and squares are already in the literature in [3] and [5] respectively. However to the knowledge of the author these identities are missing for pentagons and higher loops. The results are

$$\begin{array}{c} \text{2} \\ | \\ \text{---} \triangle \text{---} \\ / \quad \backslash \\ \text{1} \quad \text{3} \end{array} = \left( \frac{N}{2} - \frac{6}{N} \right) \begin{array}{c} \text{2} \\ | \\ \text{---} \text{---} \text{---} \\ / \quad \backslash \\ \text{1} \quad \text{3} \end{array} ,$$

$$\begin{aligned}
\text{Diagram 1} &= (1 - \frac{4}{N^2}) \left( \begin{array}{c} 24 \\ | \\ 13 \end{array} + \begin{array}{c} 43 \\ | \\ 12 \end{array} \right) + (\frac{N}{4} - \frac{4}{N}) \left( \begin{array}{c} 43 \\ | \\ 12 \end{array} + \begin{array}{c} 23 \\ | \\ 14 \end{array} \right) - \frac{N}{4} \begin{array}{c} 32 \\ | \\ 14 \end{array} , \\
\text{Diagram 2} &= (\frac{1}{2} - \frac{6}{N^2}) \left( \begin{array}{c} 35 \\ | \\ 12 \end{array} + \begin{array}{c} 53 \\ | \\ 12 \end{array} \right) - \frac{1}{N} \left( \begin{array}{c} 51 \\ | \\ 43 \end{array} + \begin{array}{c} 31 \\ | \\ 45 \end{array} + \begin{array}{c} 32 \\ | \\ 15 \end{array} + \begin{array}{c} 31 \\ | \\ 25 \end{array} \right) + \\
&\quad + \frac{1}{2} \left( \begin{array}{c} 35 \\ | \\ 14 \end{array} + \begin{array}{c} 43 \\ | \\ 15 \end{array} + \begin{array}{c} 45 \\ | \\ 23 \end{array} - \begin{array}{c} 21 \\ | \\ 34 \end{array} \right) .
\end{aligned}$$

The last identity in standard notation is

$$\begin{aligned}
Tr(D_{i_1} D_{i_2} D_{i_3} D_{i_4} D_{i_5}) &= (\frac{1}{2} - \frac{6}{N}) (d_{i_1 i_2 i_3} \delta_{i_4 i_5} + d_{i_1 i_2 i_5} \delta_{i_4 i_3}) + \\
&\quad - \frac{1}{N} (d_{i_3 i_4 k} d_{k i_2 l} d_{l i_1 i_5} + d_{i_4 i_5 k} d_{k i_2 l} d_{l i_1 i_3} + d_{i_1 i_5 k} d_{k i_4 l} d_{l i_2 i_3} - d_{i_2 i_5 k} d_{k i_4 l} d_{l i_1 i_3}) + \\
&\quad + \frac{1}{2} (Tr(D_{i_2} D_{i_1} F_{i_4} F_k) d_{k i_5 i_3} + Tr(D_{i_1} D_{i_2} D_{i_5} D_k) d_{k i_3 i_4} + Tr(D_{i_1} D_{i_2} D_{i_3} D_k) d_{k i_5 i_4} - Tr(D_{i_3} D_{i_4} D_{i_5} F_k) f_{k i_1 i_2})
\end{aligned}$$

It should be noted that all these identities have been verified in Mathematica with perfect agreement.

## 5 Summary

The aim of this paper was to give a systematic approach to compute loop tensors. The reason of doing so lies in the analysis of systems based on  $su(N)$  group. In fact the author came across this problem while studying supersymmetric Yang-Mills quantum mechanics for arbitrary  $N$  and large  $N$  limit [7,8]. This issues will be published elsewhere. The method agrees with recent results [2] where the problem was solved via characteristic equation for  $F$  matrices. Let us note that it is a laborious task to obtain this equation for arbitrary  $su(N)$  therefore a big loop diagram for large  $N$  is in general difficult to reduce. In diagrammatic approach this problem does not exist since we make no use of characteristic equation. Indeed lemmas presented here are so simple that one could write a computer program for arbitrary loop reduction. What is even more remarkable is that the derivation of our result is based only on Jacobi identities and multiplication law (1). We did not use the relations derived by Sudbery [1] although it is evident that one may contract his formulas with eg.  $d_{ijk}$  providing a constraint on a d loop.

The diagrammatic method may be applied to arbitrary Lie algebra. However since the multiplication rule (1) is different in other cases than  $su(N)$  we expect the conclusions to be different. Indeed in  $g_2$  case the situation is so different that the simplest triangle loop is not proportional to  $d_{ijk}$  tensor [6].

Finally let us note that the method [2] gives no information about lower degree traces (eg.  $Tr(F^4), Tr(F^6), Tr(F^8), Tr(F^{10})$  in  $su(5)$  cannot be written as polynomials in lower degree traces). One may however apply different arguments [5] to derive formulae for four-fold traces. Our results also agree with them. Unfortunately these arguments get more complicated while analyzing bigger loops.

## 6 Acknowledgments

I thank J. Wosiek and A. Macfarlane for discussions and encouragement. This work was supported by the Polish Committee for Scientific Research under grant no. PB 1P03B 02427 (2004-2007).

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